arXiv:gr-qc/0306111v2 7 Apr 2004

Non-singular black-holes on the brane

S. Shankaranarayanan *

DCTD, University of Azores, 9500 Ponta Delgada, Portugal.
Email: shanki@ictp.trieste.it

Naresh Dadhich

Inter University Center for Astronomy and Astrophysics,
Post Bag 4, Ganeshkhind, Pune 411 007, India.
Email: nkd@iucaa.ernet.in

We investigate the possibility of obtaining non-singular black-hole solutions in the brane world model by solving the effective field equations for the induced metric on the brane. The Reissner-Nordstrom solution on the brane was obtained by Dadhich et al by imposing the null energy condition on the 3-brane for a bulk having non zero Weyl curvature. In this work, we relax the condition of vanishing scalar curvature R, however, retaining the null condition. We have shown that it is possible to obtain class of static non-singular spherically symmetric brane space-times admitting horizon. We obtain one such class of solution which is a regular version of the Reissner-Nordstrom solution in the standard general relativity.

Keywords: black holes, extra dimensions, gravity

1. Introduction

General relativity provides an accurate description of a wide range of classical gravitational phenomena. However, the description breaks down at the space-time singularities. According to the Penrose-Hawking theorems [1] (see also Ref. 2), the manifolds arising as solutions of the Einstein field equation in general relativity (GR) are, in general, geodesically incomplete and thereby indicating the occurrence of space-time singularities like the big-bang singularity in the Universe's distant past or the singularity at the center of a black-hole. The two key assumptions of the singularity theorems are: (i) the matter must satisfy strong energy condition ($\rho \geq 0, \rho + \Sigma_i p_i \geq 0$; SEC) and (ii) space-time must satisfy appropriate causality conditions.

---

*Present Address: HEP Group, International Centre for Theoretical Physics, Strada costiera 11, 34100 Trieste, Italy.

It is now a well accepted fact that a proper understanding of these singularities or their avoidance may require new physics [3]. There have been various attempts in the literature, in this direction, for example the scalar-tensor theories of gravity in four (and other) dimensions [4] and the higher derivative gravity [5]. For the black hole singularity, the focus is two fold: (i) the *internal structure of black-holes*, the issues relating to mass-energy singularity and non-unitary black-hole dynamics, and (ii) to understand the final state of the gravitational collapse from some initial configurations. There has been quite a vigorous activity on the latter aspect for examining the cosmic censorship conjecture (see for a recent review, see Ref. 6); i.e. is the end product black hole covering the singularity or the singularity naked? Unfortunately this study throws no light on the former aspect, the nature of the singularity.

Besides these approaches there have been attempts to look for non-singular black-hole solutions within the framework of general relativity relaxing the energy conditions on the matter fields [7]. As mentioned earlier, the singularity theorems require the stress tensor of the matter to satisfy SEC. The attempts in this direction have been to look for black-hole solutions in which the matter fields satisfy weak energy condition ($\rho \geq 0, \rho + p_i \geq 0$; WEC). [Note that, from the energy-conservation theorem [1], it can be shown that the regular black-holes can not exist for the matter satisfying dominant energy condition. Matter fields satisfying weak energy condition implies null energy condition.]

The idea of replacing the black-hole singularities by non-singular vacuum cores traces back to Bardeen 8. In this model, Bardeen assumed a 4-D spherically symmetric line-element of the form

$$ds^2 = \left(1 - \frac{2R_g(r)}{r}\right) dt^2 - \left(1 - \frac{2R_g(r)}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \qquad (1)$$

where

$$R_g(r) = \frac{M\,r^3}{2(r^2 + q^2)^{3/2}} \qquad (2)$$

where M is the mass of the black-hole and q is the charge. For $q^2 < (16/27)M^2$, the line-element has an event horizon. It was also shown that the space-time satisfies WEC. Various other authors have also proposed regular black-hole solutions (for an incomplete list of references, please see Ref. 7).

Recently, Dymnikova 9 has obtained a regular exact solution (1) to the Einstein equation, where

$$R_g(r) = \frac{r_g}{2}[1 - \exp(-r^3/r_1{}^3)];$$

$$r_1 = (r_0{}^2 r_g)^{1/3}; \quad r_0{}^2 = \frac{3}{\Lambda} = l^2; \quad r_g = 2M, \qquad (3)$$

Λ is related to the positive cosmological constant. The space-time satisfies WEC. The above line-element has the following interesting properties: (i) In the limit of

$r \to 0$, the line-element goes over to a de Sitter (ii) Asymptotically, the metric goes over to a Schwarzschild (iii) the metric has two horizons – event and cosmological.

The main aim of this study is to investigate non-singular black-hole solutions in the large extra dimensional model, commonly known as the brane world models. In the recent years, there has been a renewed interest in models with extra dimensions in which the standard matter fields are confined to the four-dimensional world (viewed as an infinite hyper-surface – brane) embedded in a higher dimensional space-time (AdS bulk) where *only* gravity can propagate 10, 11, 12. In these models, 5-D bulk metric is non-factorizable, and the small value of the true five-dimensional Planck mass is related to its large effective four-dimensional value by the large warp of the five-dimensional bulk. One of the features of the brane models is that it reproduces Newtonian potential with higher order corrections. These corrections contribute significantly in the large curvature limits. (For a review on brane cosmology see, for instance, Ref. 13.)

Gravitational collapse on the brane has been studied by many authors (for an incomplete list of references please see Refs. 15, 14). In Ref. 14, the authors obtain an exact black-hole solution of the effective Einstein equation on the brane under the condition that the bulk has non zero Weyl curvature and the brane space-time satisfies the null energy condition. The solution is given by the usual Reissner-Nordstrom (RN) metric where the charge parameter is thought of as a *tidal charge* arising from the projection of the free gravity (the Weyl curvature) of the bulk onto the brane. RN metric has thus been interpreted as describing a black hole on the brane where electric charge's role is taken over by the tidal charge and it can be thought of as the analogue of the Schwarzschild solution on the brane. The tidal charge like the RN electric charge would generate $1/r^2$ term in the potential while the high energy modification to the Newtonian potential cannot be any stronger than $1/r^3$ [11,12]. The cause for this disagreement is the presence of tidal charge which is the measure of the bulk Weyl curvature. The main drawback of the solution is that we do not know the corresponding bulk solution. It is however agreed that RN metric is a good approximation to a black hole on the brane near the horizon [16]. This solution however has a singularity at the center.

In this work, we would like to look for a non-singular form of the brane black-hole solution obtained in Ref. 14 by solving the effective field equations for the induced metric on the brane. As in many other works, we focus only on the brane equation. That would mean looking for non-singular version of the usual RN metric. We shall therefore obtain non-singular RN black-hole solution in a D-dimensional spherically symmetric space-time which directly corresponds to D-brane non-singular black-hole in $(D + 1)$-dimensional bulk.

The RN solution [14] on the brane was obtained by imposing the null energy condition on the 3-brane and assuming that the bulk has non zero Weyl curvature. This gave the RN metric as the unique exact solution of the brane equation. For obtaining non-singular solution, we have to relax one of the conditions and so we

drop the condition of vanishing scalar curvature R and however retain the null condition. The latter will ensure occurrence of horizon which is however required as we wish to find non-singular brane black hole solution. It turns out that it is possible to obtain a class of static non-singular spherically symmetric brane space-times admitting horizon.

The rest of the paper is organized as follows: In section (2), we obtain a non-singular RN solution in D-dimensional spherically symmetric space-time and discuss its properties. In section (3), we discuss the implications of the non-singular RN solution in the context of the brane-world model and present our conclusions.

2. Non-singular RN black-hole in a $D-$ dimensional space-time

The line-element of a D-dimensional spherically symmetric space-time is given by

$$ds^2 = g(r)dt^2 - f(r)dr^2 - r^2 d\Omega^2 \,. \tag{4}$$

where $d\Omega^2$ is the $(D-2)$-dimensional angular line-element. If we assume that the stress tensors satisfy the equation of state of an anisotropic perfect fluid, we then have

$$T_t^t = T_r^r; \quad T_{\theta_1}^{\theta_1} = T_{\theta_2}^{\theta_2} = \cdots = T_{\theta_{D-2}}^{\theta_{D-2}}\,, \tag{5}$$

which implies $f(r) = 1/g(r)$. Under this assumption, Einstein equations of motion are

$$8\pi T_t^t = 8\pi T_r^r = -(D-2)\frac{g'(r)}{2r} + \frac{(D-2)(D-3)}{2}\left(\frac{1-g(r)}{r^2}\right) \tag{6}$$

$$8\pi T_{\theta_i}^{\theta_i} = -\frac{g''(r)}{2} - (D-3)\frac{g'(r)}{r} + \frac{(D-4)(D-3)}{2}\left(\frac{1-g(r)}{r^2}\right)$$

$$= -\frac{\nabla_r^2 g(r)}{2} + (D-4)\left[-\frac{g'(r)}{2r} + \frac{(D-3)}{2}\left(\frac{1-g(r)}{r^2}\right)\right] \tag{7}$$

where the prime denotes differentiation with respect to r and i runs from 1 to $(D-2)$. Integrating Eq. (6), we get

$$g(r) = 1 - \frac{4R_g(r)}{(D-2)r^{D-3}}\,, \tag{8}$$

where

$$R_g(r) = 4\pi \int_0^r \rho(x)x^{D-2}dx\,. \tag{9}$$

From the above expression, it is straightforward to see that different choices of ρ will lead to different forms of $R_g(r)$. As discussed in Introduction, our aim is to look for a D-dimensional non-singular charged black-hole solution satisfying Einstein's equations. Let us consider the following mass and charge distributions

$$\rho_{\text{matter}}(r) = c_1 \exp[-c_2 r^{D-1}]$$

$$\rho_{\text{charge}}(r) = \frac{c_3}{r^{2(D-2)}}\left(1 - \exp[-c_4 r^{2(D-2)}]\right) - \mathcal{Z}c_3 c_4 \exp[-c_4 r^{2(D-2)}] \tag{10}$$

where c_i's are constants which need to be determined and $\mathcal{Z}$ is a real number. The above form of density distributions are regular over all r [a]. Substituting the above density distributions in Eq. (8), we get

$$
\begin{aligned}
g(r) = 1 &- \frac{16\pi}{(D-2)(D-1)} \frac{1}{r^{D-3}} \frac{c_1}{c_2} \left[1 - \exp(-c_2 r^{D-1}) \right] \\
&+ \frac{16\pi c_3}{(D-2)(D-3)} \frac{1}{r^{2(D-3)}} \left[1 - \exp(-c_4 r^{2(D-2)}) \right] \\
&+ \frac{16\pi c_3 c_4^{1/[2(D-2)]}}{(D-2) r^{D-3}} \left[\frac{\mathcal{Z}}{2(D-2)} - \frac{1}{(D-3)} \right] \left(\Gamma \left[\frac{D-1}{2(D-2)}, c_4 r^{2(D-2)} \right] - \Gamma \left[\frac{D-1}{2(D-2)} \right] \right)
\end{aligned}
\tag{11}
$$

where Γ is the Gamma function. The following points need to be noted regarding the above expression:

(i) The metric coefficients are regular over the whole range of r.
(ii) The choice of density in Eq. (10) satisfies WEC.
(iii) In the limit of $r \to \infty$, the above line-element has the form of $D-$dimensional Reissner-Nordstrom.
(iv) In the limit of $r \to 0$, the above line-element reduces to de Sitter or Anti-de Sitter for different choices of c_i's. In the standard GR, the central core can only be de Sitter. However, in the case of brane gravity, the central core can either be de Sitter or Anti-de Sitter. We will discuss this aspect in the next section.

In Ref. 18, the author obtained a charged black-hole solution with a charged de Sitter core. It was assumed that the stress tensor of a charged material to have a uniform charge-to-mass ratio over all r. This gave the r-dependence of the charged density to be the same as that of the r-dependence for the mass density. However, the above stress tensor does not satisfy WEC as $r \to 0$. In our case, we do not assume any such form for the charge density and as mentioned earlier, the stress tensor satisfies WEC all through.

(v) The last term in the RHS of the above expression vanishes near $r = 0$. In the limit of $r \to \infty$, the leading order term is $1/r^{D-3}$. Using this fact, and setting $\mathcal{Z} = 2(D-2)/(D-3)$ in the above expression, we get

$$
\begin{aligned}
g(r) = 1 &- \frac{16\pi}{(D-2)(D-1)} \frac{1}{r^{D-3}} \frac{c_1}{c_2} \left[1 - \exp(-c_2 r^{D-1}) \right] \\
&+ \frac{16\pi c_3}{(D-2)(D-3)} \frac{1}{r^{2(D-3)}} \left[1 - \exp(-c_4 r^{2(D-2)}) \right].
\end{aligned}
\tag{12}
$$

In the rest of this section, we will consider this reduced form for mathematical simplicity. However, the results derived can be extended to the general case of Eq. (11).

---

[a] In order to see this, we take two extreme limits: a) As $r \to \infty$ the leading order terms of the charge and matter densities go as (i) $\rho_{\text{matter}} \to 0$ and (ii) $\rho_{\text{charge}} \to c_3/r^{2(D-2)}$. b) As $r \to 0$, we have (i) $\rho_{\text{matter}} \to c_1$ and (ii) $\rho_{\text{charge}} \to (1 - \mathcal{Z}) c_3 c_4$.

The constants (c_i) in the above line-element can be determined by taking the two limits — $r \to \infty$ and $r \to 0$ — and identifying them with the mass (M), charge (Q) and the cosmological constant (Λ). We, thus, have

$$\frac{8\pi c_1}{3} - \frac{Q^2}{\alpha^4} \mp \frac{1}{l^2} = 0$$

$$c_3 = (D-2)(D-3)\frac{Q^2}{16\pi}$$

$$\pm \alpha^{(D-3)}\alpha^{(D-1)} - r_g l^2 \alpha^{(D-3)} + Q^2 l^2 = 0, \tag{13}$$

where, the lower(upper) sign corresponds to Anti(de Sitter) core, $c_2 = \alpha^{-(D-1)}$ and $c_4 = \alpha^{-2(D-2)}$. Even though, the solution to the above algebraic equations are complicated, one can in-principle solve it. In the case of $D = 4$, we have

$$\alpha = \pm \frac{A^{1/2}}{2} \pm \frac{A^{1/4}}{2}\left[\frac{2r_g l^2}{A} - 1\right]^{1/2} \tag{14}$$

where

$$A = \pm \frac{2^{7/3} l r_g^{-1/2} Q^2}{3^{1/3} B} + \frac{\sqrt{r_g l^2}B}{2^{1/3} 3^{2/3}} \tag{15}$$

$$B = \left[9\sqrt{r_g l^2} + r_g^{-3/2}\sqrt{81 l^2 r_g^4 \mp 768 Q^6}\right]^{1/3} \tag{16}$$

In the limit of $Q \to 0$, we have $\alpha^3 = l^2 r_g$. This coincides with the result of Ref. 9. As noted earlier, in the case of standard general relativity Q^2 is always non-negative and hence the center has a de Sitter core.

3. Discussions and Conclusions

In Ref. 14, the analogue of the Schwarzschild solution was obtained by solving the modified vacuum equation on the brane. The bulk space-time was taken to have non zero Weyl curvature. In order to solve the equation completely, the authors imposed null energy condition on the 3-brane. Using this condition, it was shown that RN metric is the unique exact solution on the brane. The null condition ensures occurrence of horizon which is what is required for a black hole solution, however, this condition need to hold *only* at the horizon and *not necessarily* everywhere else. This is the situation for the Einstein-Yang-Mills black-hole solutions in 4D gravity where the null energy condition holds only at the horizon [19]. Numerical black-hole solutions have been obtained in which the solution is (i) RN at the horizon and (ii) Schwarzschild asymptotically. For regularity, space-time would not (as in the Schwarzschild case) be empty and hence the scalar curvature would in general be non zero even though the stress tensor contributed by the bulk Weyl curvature would be trace free. It is then possible to obtain a class of static non-singular spherically symmetric brane space-times admitting horizon. For a non-singular black hole on the brane, it is enough to find a regular version of the

RN metric which is taken to describe a black hole on the brane. In the preceding section, we have obtained a regular version of the RN metric in D-dimensional space-time;i.e. a regular black-hole solution on $(D-1)$-brane. It would be interesting to find a generalization in which the null energy condition holds only at the horizon but not globally. The solution would however be numerical as is the case for the Einstein-Yang-Mills black hole.

Even though, the brane black-hole solution is analogous to that of the RN solution in standard general relativity, the horizon and the singularity structure of the two black-holes are quite different. It depends on the sign of Q^2. For the RN solution of the standard general relativity, $Q^2 > 0$ because electric field energy is positive and it leads to the familiar two horizons. However, for the brane black-hole solution it has been argued that Q^2 must be negative as it arises from the free gravitational field (which should have negative energy [17]) of the bulk space-time [14]. It is expected that the black-hole on the brane should in the low energy limit imply corrections to the Schwarzschild solution but its singularity and horizon structure should however remain undisturbed. This can only happen if $Q^2 < 0$.

The black hole core could have been de Sitter or anti de Sitter depending upon Q^2 being positive or negative. Since this ambiguity has been resolved in favor of $Q^2 < 0$, the black hole center would be AdS. Asymptotically it approaches RN/Schwarzschild space-time while in the interior it would go to AdS at the center. The line-element (4) with $g(r)$ given by Eq. (12) describes a regular black-hole space-time free of singularity and it admits like the Schwarzschild black-hole only one horizon. Since the requirement of regularity is not uniquely specifiable, the regular solution so obtained would not however be unique. The field is therefore quite open and it is only the physical properties that would distinguish one solution from the other.

Acknowledgments

References

1. S. W. Hawking and G. F. R. Ellis, *The large scale structure of spacetime*, Cambridge University Press, Cambridge (1973).
2. J. M. M. Senovilla, *Gen. Rela. Gravi.* **29**, 701 (1997).
3. Gary T. Horowitz and R. Myers, *Gen. Rela. Gravi.* **27**, 915 (1995).
4. R. Brandenberger, V. Mukhanov and A. Sornborger, *Phys. Rev. D* **48**, 1629 (1993); K. A. Bronnikov, Phys. Rev. D **64**, 064013 (2001) [arXiv:gr-qc/0104092]; A. Bogojevic and D. Stojkovic, Phys. Rev. D **61**, 084011 (2000); K. A. Bronnikov, Phys. Rev. D **64**, 064013 (2001) [arXiv:gr-qc/0104092]; K. A. Bronnikov and G. N. Shikin, Grav. Cosmol. **8**, 107 (2002) [arXiv:gr-qc/0109027]; R. G. Daghigh, J. I. Kapusta and Y. Hosotani, arXiv:gr-qc/0008006; D. V. Gal'tsov and J. P. Lemos, Class. Quant. Grav. **18**, 1715 (2001) [arXiv:gr-qc/0008076].

5. Bob Holdom, *Phys. Rev. D* **66**, 084010 (2002) [arXiv:hep-th/0206219]; T. Hirayama and B. Holdom, arXiv:hep-th/0303174.

6. P.S. Joshi, *Mod. Phys. Lett. A* **17**, 1067 (2002).

7. M. Mars, M. M. Martin-Prats and J. M. M. Senovilla *Class. Quant. Grav.* **13**, L51 (1996); A. Borde, Phys. Rev. D **55**, 7615 (1997) [arXiv:gr-qc/9612057]. V. P. Frolov, M. A. Markov and V. F. Mukhanov, Phys. Lett. B **216**, 272 (1989). V. P. Frolov, M. A. Markov and V. F. Mukhanov, Phys. Rev. D **41**, 383 (1990). C. Barrabes and V. Frolov, Phys. Rev. D **53**, 3215 (1996) [arXiv:hep-th/9511136].

8. J. Bardeen, *in GR5 proceedings*, Tbilisi (1968).

9. I.G. Dymnikova, *Gen. Rela. Gravi.* **24**, 235 (1992); I. Dymnikova arXiv:gr-qc/0010016; I. G. Dymnikova, A. Hasan, J. Ulbricht and J. Zhai *Gen. Rela. Gravi.* **7**, 122 (2001).

10. L. Randall and R. Sundrum, *Phys. Rev. Lett.* **83**, 3370 (1999); Merab Gogberashvili, Mod. Phys. Letts. A. **14**, 2025 (1999); N. Arkani-Hamed, S. Dimopoulos, G. Dvali and N. Kaloper, *Phys. Rev. Lett.* **84**, 586 (2000); Tetsuya Shiromizu, Kei-ichi Maeda, Misao Sasaki, *Phys. Rev. D* **62**, 024012 (2000).

11. L. Randall and R. Sundrum, *Phys. Rev. Lett.* **83**, 4690 (1999).

12. J. Garriga and T. Tanaka, *Phys. Rev. Lett.* **84**, 2778 (2000).

13. R. Maartens, arXiv:gr-qc/0101059; D. Langlois, Prog. Theor. Phys. Suppl. **148**, 181 (2003). P. Kanti, arXiv:hep-ph/0402168.

14. N. Dadhich, R. Maartens, P. Padadopoulos and V. Rezania, *Phys. Lett. B* **487**, 1 (2000) [arXiv:hep-th/0003061].

15. R. Emparan, G. T. Horowitz and R. C. Myers, *JHEP* **0001**, 007 (2000); R. Emparan, G. T. Horowitz and R. C. Myers, JHEP **0001**, 021 (2000); A. Chamblin, S. W. Hawking and H. S. Reall, *Phys. Rev. D* **61**, 065007 (2000); R. Emparan, G. T. Horowitz and R. C. Myers, *Phys. Rev. Lett.* **85**, 499 (2000); N. Dadhich, *Phys. Lett. B* **492**, 397 (2000); C. Germani and R. Maartens, *Phys. Rev. D* **64**, 124010 (2001); T. Wiseman, *Phys. Rev. D* **65**, 124007 (2002); *Class. Quant. Grav.* **19**, 3083 (2002); N. Dadhich and S. G. Ghosh, *Phys. Lett. B* **518**, 1 (2001); G. Kofinas, E. Papantonopoulos and I. Pappa, Phys. Rev. D **66**, 104014 (2002); Phys. Rev. D **66**, 104028, (2002); M. Bruni, C. Germani and R. Maartens, *Phys. Rev. Lett.* **87**, 231302 (2001); M. Govender and N. Dadhich, *Phys. Lett. B* **538**, 233 (2002); Rui Neves and Cenalo Vaz, *Phys. Rev. D* **66**, 124002 (2002); P. Kanti and K. Tamvakis, Phys. Rev. **D65**, 084010 (2002); P. Kanti, I. Olasagasti and K. Tamvakis, Phys. Rev. D **68**, 124001 (2003) [arXiv:hep-th/0307201]; H. Kudoh, T. Tanaka, T. Nakamura, arXiv:gr-qc/0301089; Rui Neves and Cenalo Vaz, `hep-th/0302030`; M. Cavaglia, arXiv: hep-ph/0305256, H. Kudoh, arXiv: hep-th/0306067.

16. A Chamblin, H S Reall, H S Shinkai and T Shiromizu, *Phys. Rev. D* **63**, 064015 (2001).

17. N. Dadhich, *Phys. Lett. B* **492**, 357 (2000).

18. W. F. Kao, arXiv: hep-th/0009049.

19. R. Bartnik and J. Mckinnon, Phys. Rev. Lett. **61**, 141 (1988). A. Corichi, U. Nucamendi and D. Sudarsky, Phys. Rev. D **64**, 107501 (2001) [arXiv:gr-qc/0106084]; A. Corichi and D. Sudarsky, Phys. Rev. D **61**, 101501 (2000) [arXiv:gr-qc/9912032]; F. Finster, J. Smoller and S. T. Yau, J. Math. Phys. **41**, 3943 (2000) [arXiv:gr-qc/0001064]. A. Ashtekar, A. Corichi and D. Sudarsky, Class. Quant. Grav. **18**, 919 (2001) [arXiv:gr-qc/0011081]; E. Radu, Phys. Rev. D **65**, 044005 (2002) [arXiv:gr-qc/0109015]; M. S. Volkov and D. V. Gal'tsov, Phys. Rept. **319**, 1 (1999) [arXiv:hep-th/9810070]; D. V. Gal'tsov, arXiv:gr-qc/9808002.